Instanton counting and wall-crossing formula for Donaldson invariants

Hiraku Nakajima

based on joint works with Kota Yoshioka,
Lothar Göttsche, Takuro Mochizuki

Gauge Theory and Representation Theory

Nov. 29, 2007 Princeton

1988 Witten

Donaldson invariants = correlation functions in
twisted 4D, N=2 SUSY YM theory

1994 Seiberg-Witten

solution of (ordinary) 4D N=2 SUSY YM on $\mathbb{R}^4$
via a family of elliptic curves ⇝ Seiberg-Witten invariants
("u-plane")

1997 Moore-Witten

analysis of Donaldson invariants via SW solution (u-plane integral)
e.g.
- blowup formula of Fintushel-Stern
- wall-crossing formula of Göttsche
- ⋮

in terms of theta functions

However, SW solution and MW argument
were not mathematically justified at that time.

Q. Understand mathematically

1) What does $N=2$ SYM on $\mathbb{R}^4$ compute?
2) Why it is useful to study Donaldson invariants?

Answer to Q1 was given much later ....

2002 Nekrasov
mathematically rigorous definition of the (deformed) partition function of
$N=2$ SUSY YM on $\mathbb{R}^4$.

2003 N-Yoshioka, Nekrasov-Okounkov, Braverman-Etingof
rigorous derivation of SW solution from Nekrasov's definition

Today's Goal :

Answer to Q2.

Also deformed partition function is recovered quite naturally.

(i.e, refinement of the answer to Q1)

Method is applicable to other situations in 4D gauge theory

e.g. • K-theoretic invariants

• (virtual) Euler #, (virtual) χ_y-genus

(special cases of K-theoretic invariants)

$\sim$ related to BPS states (Moore's talk)

Some of features should also appear in the wall-crossing of DT invariants in CY 3-folds.

Plan of the talk :

1) A review of the definition of Donaldson's invariants
2) A review of the wall-crossing under the change of the stability condition
cf. Thomas & Moore's talks
3) Wall-crossing formula in terms of products of Hilbert schemes
4) Nekrasov's deformed partition function
5) Taking the classical limit to recover Göttsche's wall-crossing formula

Today :
$\left(\begin{array}{l} X : \text{smooth projective surface } /\mathbb{C} \\ H : \text{ample line bundle} \end{array} \right.$

Although Donaldson invariants are C^∞-invariants of 4-mfds, our main techniques (virtual localization, Hilb. scheme) are available only in an algebro-geometric situation.
So we start with algebro-geometric setting from the beginning.

Def. A coherent sheaf E on X is (Gieseker-)semistable w.r.t. H

$\Longleftrightarrow$ $\forall S$: subsheaf of E s.t. $0 < \operatorname{rank} S < \operatorname{rank} E$

$$\frac{\chi(S(mH))}{\operatorname{rank} S} \underset{\leq}{<} \frac{\chi(E(mH))}{\operatorname{rank} E} \quad \text{for } m \gg 0$$

χ = holomorphic Euler characteristic

Def.

$M_H \equiv M_H(r, c_1, c_2)$: moduli scheme of semistable sheaves (projective variety)

$\cup$

M_H^s : moduli of stable sheaves

Deformation theory is controlled by $\mathrm{Ext}^i_0(E,E)$

$\mathrm{Ext}^0_0(E,E)$: automorphism

$\mathrm{Ext}^1_0(E,E)$: infinitesimal deformation

$\mathrm{Ext}^2_0(E,E)$: obstruction

stable $\Rightarrow$ $\mathrm{Ext}^0_0(E,E) = \mathrm{Hom}_0(E,E) = 0$

<u>Fact</u> (Donaldson, Friedman, Zuo, Gieseker-Li, O'Grady, ...)

r, c_1, H : fix

If $c_2 \gg 0$ $\Rightarrow$ $\mathrm{Ext}^2_0(E,E) = 0$ except for $E \in$ lower dimensional subvariety

$\therefore$ M_H is of expected dimension for $c_2 \gg 0$.

(This is the reason why we don't need to consider the virtual fundamental class in Donaldson theory.)

Assume $M_H = M_H^s$ (e.g. $GCD(r, (c_1(E).H)) = 1$) for simplicity.

$\mathcal{E}$: universal sheaf over $X \times M_H$

$\mu_p : H_*(X) \longrightarrow H^*(M_H)$; $\alpha \longmapsto (-1)^p \left[ch(\mathcal{E}) \cdot e^{-\frac{c_1(\mathcal{E})}{r}} \right]_{p+1} / \alpha$

(generalised) Donaldson invariant : (fix r, c_1)

$$\boxed{\sum_{c_2} \Lambda^{\dim_{\mathbb{C}} M_H(r,c_1,c_2)} \int_{M_H(r,c_1,c_2)} \exp\Big(\sum_{p=1}^{\infty} \mu_p(\alpha_p)\Big)}$$

NB ordinary Donaldson invariants : $r=2$, $p=1$ case

The invariants, a priori, depend on H.

- $r=2, p=1$
  $\Rightarrow$ The above $=$ ordinary Donaldson inv.

  if further $p_g > 0$ ($\Leftrightarrow b^+ \geqq 3$)
  $\Rightarrow$ independent of H

On the other hand, if $p_g = 0$ $\Longrightarrow$ $\boxed{\text{Wall-crossing}}$

<u>Rem.</u> independence when $p_g > 0$ was shown by Mochizuki.

○ Review of Wall-crossing

H_+, H_- : ample line bundles

s.t. $M_{H_+} = M^s_{H_+}$, $M_{H_-} = M^s_{H_-}$

Suppose E_+ : H_+-stable , not H_--stable .

$\therefore\ \exists\, S \subset E_+$: subsheaf

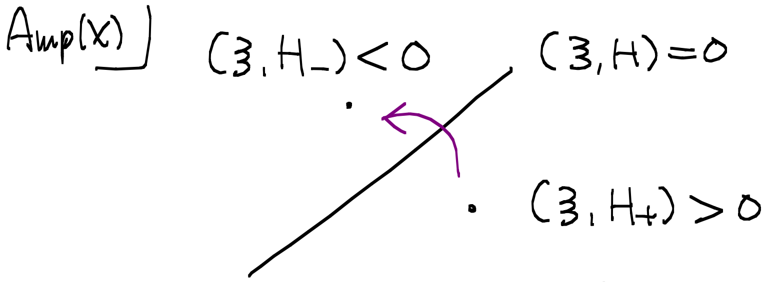
s.t. $\cdot\ \dfrac{\chi(S(mH_+))}{\mathrm{rank}\, S} < \dfrac{\chi(E_+(mH_+))}{\mathrm{rank}\, E_+}$

$\cdot\ \dfrac{\chi(S(mH_-))}{\mathrm{rank}\, S} > \dfrac{\chi(E_+(mH_-))}{\mathrm{rank}\, E_+}$

$$\Longrightarrow \begin{aligned} &\big(c_1(E_+)\cdot \mathrm{rank}\, S - c_1(S)\cdot \mathrm{rank}\, E_+ ,\ H_+\big) > 0 \\ &\big(\underbrace{\qquad\qquad\qquad \text{''} \qquad\qquad\qquad}_{\|}\ ,\ H_-\big) < 0 \end{aligned}$$

Let ξ $\in NS(X)$.

Then ξ defines a wall in $\mathrm{Amp}(X)$ (= ample cone of X)



From now we fix ξ and assume there are no other walls between H_+ and H_-.

Let $Q = E_+/S$.

Then
$$0 \to S \to E_+ \longrightarrow Q \to 0$$

defines a class in $[E_+] \in \mathbb{P}(\mathrm{Ext}^1(Q, S))$.

We can exchange $Q \leftrightarrow S$

$$0 \to Q \to E_- \to S \to 0 \quad (\leftrightarrow [E_-] \in \mathbb{P}(\mathrm{Ext}^1(S,Q)))$$

gives a H_--stable, H_+-unstable sheaf E_-.

We can now move S, Q in moduli spaces of <u>lower</u> <u>ranks</u>.

So a "rough" picture of the change of moduli spaces :

$$\begin{tikzcd}
M_{H_+} \arrow[rr, dashed, leftrightarrow] & & M_{H_-} \\
\mathbb{P}(\mathrm{Ext}^1(\mathcal{Q},\mathcal{S})) \arrow[u, hook] \arrow[dr] & & \mathbb{P}(\mathrm{Ext}^1(\mathcal{S},\mathcal{Q})) \arrow[u, hook] \arrow[dl] \\
& M_H(r',c_1',c_2') \times M_H(r'',c_1'',c_2'') &
\end{tikzcd}$$

$\mathcal{S}, \mathcal{Q}$: universal sheaves

projective space bundles

<u>Rem</u>. In DT / stable pair wall-crossing,
H is no longer ample line bundle.
In fact, $\xi = \mathbb{C}_{pt}$.

A technical comment, which I suggest to ignore at the first reading.

In general, a) $Ext^1(S, Q)$ is not a vector bundle.

b) $M_H(r', c_1', c_2') \neq M_H^s(r', c_1', c_2')$ as H is on the wall.

a) (T. Mochizuki)

Virtual fundamental classes on

- moduli of stable sheaves (in fact, stable pairs)
- master spaces, connecting two GIT quotients w.r.t. two different polarizations

$$\begin{tikzcd}
& \mathbb{P}(L_1 \oplus L_2)/\!/G \arrow[dl] \arrow[dr] & \mathbb{C}^* \arrow[l, bend right] \\
X/\!/_{L_1} G & & X/\!/_{L_2} G
\end{tikzcd}$$

b) (T. Mochizuki)

Work recursively. ---> to be explained later, if I have time.

To make life easy, we assume (as in [GNY]),

& $\cdot$ rank $= 2$ $(\Rightarrow r' = r'' = 1 \Rightarrow M_H({r' \atop r''}, \cdots) \cong$ Hilbert scheme of points

$\cdot$ the wall is good $(\Rightarrow \operatorname{Ext}^0, \operatorname{Ext}^2 = 0$ along exceptional loci$)$

Then the above picture is precise, and we do not need virtual fundamental classes.

We get wall-crossing terms

$$= \boxed{\sideset{}{'}\sum_{c_2', c_2''} \operatorname*{Res}_{a=\infty} \int_{\underset{\underset{\mathcal{S}}{\vdots}}{M(1,c_1',c_2')} \times \underset{\underset{\mathcal{Q}}{\vdots}}{M(1,c_1'',c_2'')}} \frac{\exp(\sum \mu_p(\mathcal{E} \text{ replaced by } \mathcal{Q}e^a \oplus \mathcal{S}e^{-a}))}{e(\operatorname{Ext}^1(\mathcal{Q}e^a, \mathcal{S}e^{-a}))e(\operatorname{Ext}^1(\mathcal{S}e^{-a}, \mathcal{Q}e^a))}}$$

$$\left(\begin{array}{l} c_1' = c_1 - \frac{3}{2} \\ c_1'' = c_1 + \frac{3}{2} \end{array}\right.$$

e^a : formal variable

If the wall is not good, we need to use the virt. fund. class.

The corrected answer is simple to guess :

Replace Ext^1 by $-\operatorname{Ext}^0 + \operatorname{Ext}^1 - \operatorname{Ext}^2$.

Q. How to compute this intersection product on $\mathrm{Hilb} \times \mathrm{Hilb}$?

Step1°. We prove the above is "universal" w.r.t. X, ξ, α_p, i.e.
it depends only on various intersection pairing between
products of $c_1(X)$, $c_2(X)$, ξ, α_p. (cf. Ellingsrud-Göttsche
-Lehn)

Step2°. By step1°, we may assume
X = projective toric surface (e.g. $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$)
α_p : equivariant cohom. class, ξ : equiv. line b'dle

Then we use the Atiyah-Bott-Lefschetz formula.
$\mathrm{Hilb}^n X \curvearrowleft T^2$ has isolated fixed points,
so we have a combinatorial expression.

Rem. $M_H \curvearrowleft T^2$ does not have isolated fixed pts in general.
So a direct application of ABL formula to M_H is
not useful.

Step3°. We still need to evaluate the combinatorial expression.
⇝ Nekrasov's deformed partition function.
← various approaches

ABL residue formula

M : cpt cpx mfd $\curvearrowleft T$, isolated fixed pts

$\alpha \in H^*_T(M)$

$$\Rightarrow \int_M \alpha = \sum_{p \in M^T} \frac{\alpha_p}{e_T(T_pM)} \qquad \alpha_p = i_p^* \alpha$$

$$i_p : \{p\} \hookrightarrow M$$

X = toric surface

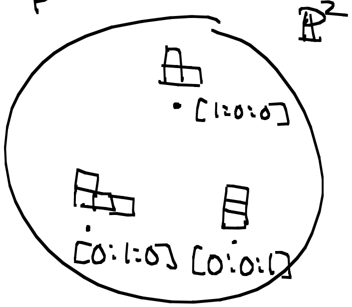
$X^T = \{p_1, \cdots, p_\chi\}$ where $\chi = \# X^T$ = Euler # of X

$M = \mathrm{Hilb}^n X$ = Hilbert scheme of n points on X

$$\Longrightarrow \quad M^T = \{ Z^1 \cup \dots \cup Z^\chi \mid Z^i : \text{supported at the fixed pt } p^i \}$$

$\updownarrow$

monomial ideal in toric coord. around p_i $\longleftrightarrow$ Young diagram λ_i

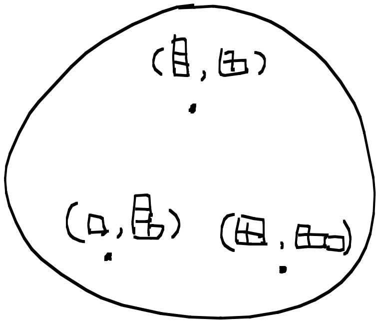


In our application, we compute $\int_{\mathrm{Hilb}^n \times \mathrm{Hilb}^m}$

So fixed pts $\longleftrightarrow$ $(\underbrace{Z_1^1, Z_2^1}_{\substack{\text{pair of} \\ \text{0-dim. subschemes}}}, \underbrace{Z_1^2, Z_2^2}, \cdots, Z_1^\chi, Z_2^\chi)$

$\longleftrightarrow$ $(\vec{\lambda}^1, \vec{\lambda}^2, \cdots, \vec{\lambda}^\chi)$

$\vec{\lambda}^i = (\lambda_1^i, \lambda_2^i)$: pair of Young diagram



$\therefore$ wall crossing term

$$= \operatorname*{Res}_{a=\infty} \lim_{\varepsilon_1, \varepsilon_2 \to 0} \sum_{\substack{p \in \mathrm{Hilb}^T \times \mathrm{Hilb}^T \\ \updownarrow \\ \text{fixed pt } (\vec{\lambda}^1, \cdots \vec{\lambda}^\chi)}} \frac{\exp(\sum \mu_p(\ \mathcal{E} \text{ replaced by } Qe^a \oplus \mathcal{S}e^{-a}))|_p}{e(\mathrm{Ext}^1(Qe^a, \mathcal{S}e^{-a}))|_p \, e(\mathrm{Ext}^1(\mathcal{S}e^{-a}, Qe^a))|_p \, e(T_p(\mathrm{Hilb} \times \mathrm{Hilb}))}$$

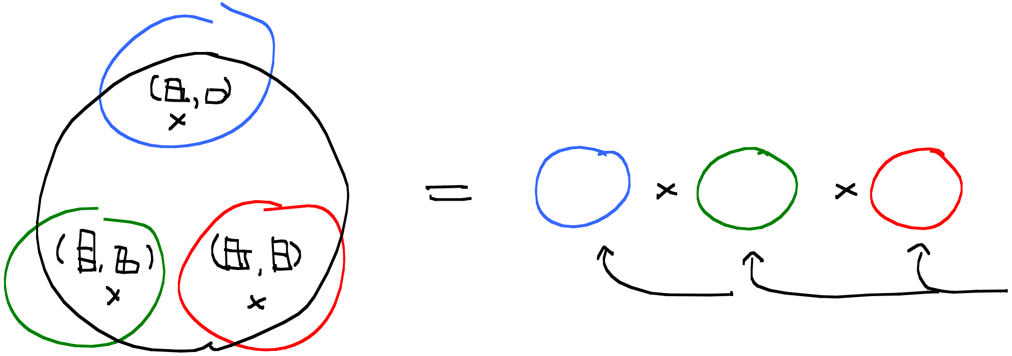
Observation (Cut the mfd X to "local pieces" !)

① $$\sum_{(\vec{\lambda}^1,\dots,\vec{\lambda}^{\chi})} \cdots\cdots = \prod_{i=1}^{\chi}\left(\sum_{\vec{\lambda}^i} \cdots\cdots\right)$$

↑ local contribution of the fixed pt p_i to the wall-crossing formula.

② local contribution is "universal", i.e.
it depends on · $w(x_i), w(y_i)$: weights of toric coord's
· $\alpha_p|_{p_i}$, $\xi|_{p_i}$

i.e.,



related by linear change of variables

Thus it is enough to calculate the local contribution of $0 \in \mathbb{C}^2 \leftarrow T^2$

$$(x, y) \mapsto (e^{\varepsilon_1}x, e^{\varepsilon_2}y)$$

Th. This local contribution is equal to

Nekrasov deformed partition function.

i.e. given by equivariant integration over framed moduli space of rk 2 torsion-free sheaves on $\mathbb{P}^2$

= (resolution of) framed moduli space of instantons on $\mathbb{R}^4$

$$M(2,n) = \{(E,\Phi) \mid E : \text{torsion free sheaf on } \mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty \}$$

$$\Phi : E|_{\ell_\infty} \to \mathcal{O}_{\ell_\infty}^{\oplus 2}$$

$$\curvearrowleft \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$$

$(x,y) \mapsto (e^{\varepsilon_1}x, e^{\varepsilon_2}y)$

change of the framing by $\begin{bmatrix} e^a & 0 \\ 0 & e^{-a} \end{bmatrix}$

deformed partition function

- instanton part

$$Z^{inst} = \sum_{n=0}^{\infty} \Lambda^{4n} \int_{M(2,n)} \exp\Big(\sum_{p=1}^{\infty} (-1)^p \tau_p \, ch_{p+1}(\mathcal{E})/[\mathbb{R}^4]\Big)$$

$$= \sum_{n=0}^{\infty} \Lambda^{4n} \sum_{p \in M(2,n)^{\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*}} \frac{\exp(\cdots\cdots)|_p}{e(T_p M(2,n))}$$

$$\in \mathbb{Q}(\varepsilon_1, \varepsilon_2, a)[[\tau_1, \tau_2, \cdots\cdots, \Lambda]]$$

- perturbation part

$Z^{pert.}$ = some explicit function

$$Z(\varepsilon_1, \varepsilon_2, a, \tau_1, \cdots\cdots; \Lambda) \equiv Z := Z^{pert} \times Z^{inst}$$

If we identify $M(2,n)^{\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*} = \{\vec{\lambda} = (\lambda_1, \lambda_2) \mid |\lambda_1| + |\lambda_2| = n\}$,
then Theorem is trivial.

Recall :

$$\frac{\exp\left(\sum \mu_p(\mathcal{E} \text{ replaced by } Qe^{a}\oplus Se^{-a})\right)\Big|_p}{e(\mathrm{Ext}^1(Qe^{a}, Se^{-a}))\big|_p\, e(\mathrm{Ext}^1(Se^{-a}, Qe^{a}))\big|_p\, e(T_p(\mathrm{Hilb}\times\mathrm{Hilb}))}$$

On the other hand,

$$T_{\underset{I_{Z_1}\oplus I_{Z_2}}{\underset{\shortparallel}{\textcircled{p}}}}M(2,n) \cong \mathrm{Ext}^1(I_{Z_1}, I_{Z_2}(-\ell_\infty)) \oplus \mathrm{Ext}^1(I_{Z_2}, I_{Z_1}(-\ell_\infty))$$
$$\oplus\, \mathrm{Ext}^1(I_{Z_1}, I_{Z_1}(-\ell_\infty)) \oplus \mathrm{Ext}^1(I_{Z_2}, I_{Z_2}(-\ell_\infty))$$

Rem.

perturbation part = local contribution to with all $\lambda^i_\alpha = \phi$

(i.e. line b'dles)

○ Taking limit $\varepsilon_1, \varepsilon_2 \longrightarrow 0$

$$\operatorname*{Res}_{a=0} \lim_{\varepsilon_1,\varepsilon_2\to 0} \prod_{i=1}^{p} Z(\underset{\underset{\varepsilon_1}{\uparrow}}{w(x_i)}, \underset{\underset{\varepsilon_2}{\uparrow}}{w(y_i)}, \underset{\underset{a}{\uparrow}}{a-\xi(p_i)}, \underset{\underset{\tau_1}{\uparrow}}{\alpha_1(p_i)}, \underset{\underset{\tau_2}{\uparrow}}{\alpha_2(p_i)}, \cdots, \Lambda)$$

From ABL formula applied to X, we have

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \sum \frac{\alpha_m(p_i)\alpha_n(p_i)}{w(x_i)w(y_i)} = \int_X \alpha_m \alpha_n \quad \text{etc}$$

$$\sum \frac{w(x_i)\, w(y_i)}{w(x_i)w(y_i)} = \underset{\text{Euler \#}}{\chi(X)}, \quad \sum \frac{w(x_i)^2 + w(y_i)^2}{3w(x_i)w(y_i)} = \underset{\text{signature}}{\sigma(X)} \quad \text{etc}$$

We substitute $\log Z = \dfrac{1}{\varepsilon_1\varepsilon_2}\left(F + (\varepsilon_1+\varepsilon_2)H + \varepsilon_1\varepsilon_2 A + \dfrac{\varepsilon_1^2+\varepsilon_2^2}{3} B + \text{higher}\right)$

$\uparrow$ Seiberg-Witten prepotential

$$= \operatorname*{Res}_{a=\infty} \exp\left(\sum_{i=1}^{\chi} \frac{1}{w(x_i)w(y_i)} \left(F(a,0,0,\cdots;\Lambda) + \frac{\partial F}{\partial a}(a,0,0,\cdots;\Lambda)\xi(p_i) + \frac{\partial F}{\partial \tau_p}(a,0,\cdots 0;\Lambda)\alpha_p(p_i) + \cdots \right)\right)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\sum \frac{1}{w(x_i)w(y_i)} = \int_X 1 = 0 \qquad \sum \frac{\xi(p_i)}{w(x_i)w(y_i)} = \int_X \xi = 0 \qquad \sum \frac{\alpha_p(p_i)}{w(x_i)w(y_i)} = \int_X \alpha_p$$

$$= \operatorname*{Res}_{a=0} \exp\Big[\frac{\partial F}{\partial \tau_p} \cdot \int_X \alpha_p + \frac{\partial^2 F}{\partial a \partial \tau_p} \cdot \int_X \xi \alpha_p + \frac{\partial^2 F}{\partial \tau_p \partial \tau_{p'}} \int_X \alpha_p \alpha_{p'}$$
$$+ \frac{\partial H}{\partial a} \int_X c_1(X) \cdot \xi + \frac{\partial H}{\partial \tau_p} \int_X c_1(X) \alpha_p$$
$$+ A \cdot \chi(X) + B \cdot \sigma(X) \Big]$$

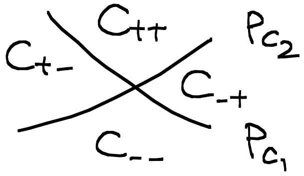
- F has been computed by various people.

- rank 2 , $p=1$ (the usual Donaldson invariants)
  $\Longrightarrow$ H, A, B are computed.
  (expected to be generalized .....)

In fact, H comes only from the perturb. part.
$\longleftrightarrow$ $c_1(X)$ appears only in the orientation of the moduli sp.
$\uparrow$
depending on the cpx str.

o Higher rank case

We should consider the wall-crossing of the wall-crossing :



$$\{(\text{inv. for } C_{++}) - (\text{inv. for } C_{+-})\}$$
$$- \{(\text{inv. for } C_{-+}) - (\text{inv. for } C_{--})\}$$

More generally, we should consider

the wall crossing of (the wall-crossing of (the wall-crossing of (the wall-crossing of (--------)))

(rk − 1 times)

The final term ☆ $= \displaystyle\sum \int_{M(1,c_1^{(1)},c_2^{(1)})\times\cdots\times M(1,c_1^{(r)},c_2^{(r)})} \frac{\text{poly in } \mu}{\prod_{i\neq j} e(-\mathrm{Ext}^{\bullet}(\mathcal{E}_i e^{a_i}, \mathcal{E}_j e^{a_j}))}$

↑ r copies of Hilb.

Final comments:

Donaldson invariants only see lower terms F, H, A, B.

Q. Meaning of higher terms?

- $\varepsilon_1+\varepsilon_2=0 \to$ higher genus GW invariants of noncpt CY 3 fold via geometric engineering

  $\varepsilon_1+\varepsilon_2\neq 0 \to$ and refined BPS invariants

- SU(N)-CS partition function of lens space (or Hopf link)

  and its homology version (à la Khovanov)

But do we have 4-dimensional interpretation?

? Donaldson invariants for families of 4-mfds?

or integrate over cycles in $X \times \mathrm{Met}/\mathrm{Diff}\,X$?